MAT337 Notes

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Contents

1 The Real Numbers

1.1 Preliminaries

Theorem. For $a, b \in \mathbb{R}$, a and b are equal iff $\forall \varepsilon > 0$, $|a - b| < \varepsilon$.

1.2 Axiom of Completeness

1.2.1 Initial definition of $\mathbb{R}$

- $\mathbb{Q} \subseteq \mathbb{R}$
- $\mathbb{R}$ is a field, so commutativity, associativity and distributivity will hold

Axiom of Completeness: Every non-empty set of real numbers that is bounded above has a supremum.

1.2.2 Lowest Upper Bound and Greatest Lower Bound

Definition. A set $A \subseteq \mathbb{R}$ is bounded above if there exists some $b \in \mathbb{R}$ such that for all $a \in A$, $a \leq b$. b is an upper bound of A.
A set $A \subseteq \mathbb{R}$ is bounded below if there exists some $b \in \mathbb{R}$ such that for all $a \in A$, $a \geq b$. b is an lower bound of A.

Definition. A number $s \in \mathbb{R}$ is the supremum of a set $A \subseteq \mathbb{R}$ if

1. s is an upper bound of A
2. If b is an upper bound of A, $s \leq b$

If so, $s = \sup(A)$

- $\inf(A)$ is defined similarly
- sup and inf are unique

Lemma. Assume $y \in \mathbb{R}$ is an upper bound of $X \subseteq \mathbb{R}$. Then, $y = \sup(X)$ iff for all $\varepsilon > 0$, there exists some $x \in X$ such that $y - \varepsilon < x$.

1.3 Consequences of Completeness

Theorem (Nested Interval Property)**.** For each $n \geq 1$, let $I_n = [a_n, b_n]$ be a closed interval such that $I_{n+1} \subseteq I_n$. Then

$$\bigcap_{n \geq 1} I_n \neq \emptyset$$

1.3.1 Density of $\mathbb{Q}$ in $\mathbb{R}$

Archimedean property

1. Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$
2. Given any $y > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y$

Definition. A set $D \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if for any two $x, y \in \mathbb{R}$ such that $x < y$, there exists some $d \in D$ such that

$$x < d < y$$

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** $\mathbb{Q}$ is dense in $\mathbb{R}$.

- $\mathbb{R} \setminus \mathbb{Q}$ is also dense in $\mathbb{R}$

1.4 Cardinality

Definition. A set X is countable if there exists a bijection $f : \mathbb{N} \to X$.

Theorem. The following are true:

1. $\mathbb{Z}$ is countable
2. $\mathbb{Q}$ is countable
3. The product of finitely many countable sets is countable.

Theorem. $\mathbb{R}$ is uncountable.

Theorem. If $A \subseteq B$ and B is countable, then A is countable, empty, or finite.

Definition. The power set $\mathscr{P}(X)$ is the set consisting of all subsets of X.

Lemma. For every set X, there exists a bijection from $\mathscr{P}(X)$ onto $\{0, 1\}^X$.

Theorem (Cantor's Theorem I)**.** $\mathscr{P}(\mathbb{N})$ is uncountable.

Theorem (Cantor's Theorem II)**.** For every X, there is no surjection from X onto $\mathscr{P}(X)$.

2 Sequences and Series

2.1 Sequences and their limits

Definition. A sequence is a function $f : \mathbb{N} \to \mathbb{R}$.

Definition. A sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, $|x - x_n| < \varepsilon$.

Definition. Given $x \in \mathbb{R}$ and $\varepsilon > 0$, the set $V_\varepsilon(x) = \{y \in \mathbb{R} : |x - y| < \varepsilon\}$ is the ε-neighbourhood around x.

Definition. A sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in \mathbb{R}$ if given any $V_\varepsilon(x)$, there exists some $N \in \mathbb{N}$ such that $x_n \in V_\varepsilon(x)$ whenever $N \in \mathbb{N}$.

Definition. A sequence that doesn't converge diverges.

Proposition. If a sequence $(x_n)_{n\in\mathbb{N}}$ converges, then so does $(|x_n|)_{n\in\mathbb{N}}$.

Proposition. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence and $(y_n)_{n\in\mathbb{N}}$ be a sequence with the following property: there exists some $R > 0$ and $N \in \mathbb{N}$ such that such that for all $n \geq N$, $|x_n - y_n| < R$. Then $(y_n)_{n\in\mathbb{N}}$ is also bounded.

2.2 Limits, Order and Algebraic Operations

Definition. A sequence $(x_n)_{n\in\mathbb{N}}$ is bounded if there exists some $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem. Every convergent sequences is bounded.

Algebraic Limit Theorem

Let $\lim a_n = a$ and $\lim b_n = b$. Then,

1. $\lim(ca_n) = ca$ for all $c \in \mathbb{R}$
2. $\lim(a_n + b_n) = a + b$
3. $\lim(a_n b_n) = ab$
4. $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ provided $b \neq 0$

Order Limit Theorem

Assume $\lim a_n = a$ and $\lim b_n = b$. Then

1. If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$
2. If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$
3. If there exists some $c \in \mathbb{R}$ such that $c \leq b_n$ or $a_n \leq c$ then $c \leq b$ or $a \leq c$

2.3 Monotone Convergence Theorem

Definition. A sequence $(a_n)_{n\in\mathbb{N}}$ is increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is monotone if either increasing or decreasing.

Monotone Convergence Theorem

If a sequence is bounded or monotonic, then it converges.

2.4 Bolzano-Weierstrass Theorem

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $n_1 < n_2 < \ldots$ be an increasing sequence of natural numbers. Then $(x_{n_1}, x_{n_2}, \ldots)$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and denoted $(x_{n_k})_{k\in\mathbb{N}}$.

Proposition. Subsequences of convergent sequences converge to the same limit as the original.

Bolzano-Weierstrass Theorem
Every bounded sequence has a convergent subsequence.

2.5 Cauchy Sequences

Definition. A sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if for every $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n, m \geq N$, $|x_n - x_m| < \varepsilon$.

Theorem (Cauchy Criterion)**.** A sequence is convergent if, and only if, it is Cauchy.

2.6 Infinite Series

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence. An infinite series is an expression of the form

$$\sum_{n\in\mathbb{N}} x_n := x_1 + \ldots + x_n + \ldots$$

The sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ is given by

$$s_n := \sum_{i\leq n} x_i = x_1 + x_2 + \ldots + x_n$$

A series converges to s if its sequence of partial sums does, i.e.: $s_n \to s$. If so, we say

$$\sum_{n\in\mathbb{N}} x_n = s$$

Theorem (Cauchy Condensation Test)**.** Suppose $(x_n)_{n\in\mathbb{N}}$ is decreasing and satisfies $x_n \geq 0$ for all $n \in \mathbb{N}$. Then

$$\sum_{n\in\mathbb{N}} x_n \text{ converges iff } \sum_{n\in\mathbb{N}} 2^n x_{2^n} \text{ does}$$

Theorem. If $\sum_{n\in\mathbb{N}} x_n = s$ and $\sum_{n\in\mathbb{N}} y_n = t$, then

1. For all $k \in \mathbb{R}$, $\sum_{n\in\mathbb{N}} kx_n = ks$
2. $\sum_{n\in\mathbb{N}} (x_n + y_n) = s + t$

Corollary. If $\sum_{n\in\mathbb{N}} x_n$ converges, then $x_n \to 0$.

Corollary (Comparison Test)**.** Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$. Then

1. If $\sum_{n\in\mathbb{N}} y_n$ converges, then so does $\sum_{n\in\mathbb{N}} x_n$
2. If $\sum_{n\in\mathbb{N}} x_n$ diverges, then so does $\sum_{n\in\mathbb{N}} y_n$

Theorem (Cauchy Criterion for Series)**.** The series $\sum_{n\in\mathbb{N}} a_n$ converges iff given $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n > m \geq N$,

$$|a_{m+1} + \ldots + a_n| < \varepsilon$$

3 Topology

3.1 Open and Closed Sets

3.1.1 Open Sets

Definition. A subset $U \subseteq \mathbb{R}$ is open if for all $x \in U$, there exists some $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq U$.

Examples:

- $\mathbb{R}$
- $\emptyset$
- Any open interval (a, b), $a, b \in \mathbb{R}$

Proposition. The following hold for open sets:

- Arbitrary unions of open sets are open
- Finite intersections of open sets are open

3.1.2 Closed Sets

Definition. A point x is a limit point of a set X if every neighbourhood of x intersects the set X at some point other than x. Equivalently, x is a limit point of X if

$$(V_\varepsilon(x) \setminus \{x\}) \cap X \neq \emptyset$$

for every $\varepsilon > 0$.

Theorem. x is a limit point of A if, and only it, $x = \lim a_n$ for some sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ satisfying $a_n \neq x$ for all $x \in \mathbb{N}$.

Definition. A point $a \in A$ is an isolated point of A if a is not a limit point of A.

Definition. A set $F \subseteq \mathbb{R}$ is closed if it contains all its limit points.

Example:

- Convergent sequences
- Closed intervals

Theorem. A set $A \subseteq \mathbb{R}$ is closed if, and only if, every Cauchy sequence contained in A has a limit that is also an element of A.

Proof. Suppose A is closed and let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence contained A. By the Cauchy criterion, (a_n) converges to some limit a. If $a_n \neq a$ for all $n \in \mathbb{N}$, then a is a limit point of A by definition, so $a \in A$. If for some n $a_n = a$, then since (a_n) is contained in A, $a \in A$.
Suppose then that every Cauchy sequence contained in A has a limit also contained in A. Let x be a limit point of A. By definition, there exists some sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ such that $a_n \neq x$ for all n and $a_n \to x$. Since this sequence is convergent, then it is also Cauchy. By assumption, this shows $x \in A$, thus A is closed. ■

Definition. The closure $\bar{X}$ of a set X is defined as X together with its limit points.

Theorem. The closure $\bar{X}$ of X is the minimal closed set including X.

Proposition. A subset of $\mathbb{R}$ is open if, and only it, its complement is closed.

Corollary. The following hold for closed sets:

- Arbitrary intersections of closed sets are closed
- Finite unions of closed sets are closed.

Proof. By DeMorgan's. ■

3.2 Compactness

Definition. A subset $A \subseteq \mathbb{R}$ is compact if every sequence in A has a subsequence that converges to some point in A.

Example

- Closed intervals

Heine-Borel Theorem
A subset K of $\mathbb{R}$ is compact if, and only if, it's closed and bounded.

Proposition. The nested intersection of non-empty compact sets is non-empty.

3.2.1 Open Covers

Definition. An open cover of a subset $X \subseteq \mathbb{R}$ is a collection of open sets $\{O_i : i \in I\}$ whose union includes X, i.e.:

$$X \subseteq \bigcup_{i \in I} O_i$$

Given an open cover of X, a finite subcover is a finite subcollection of open sets form the original open cover whose union still covers X.

Theorem. Let $K \subseteq \mathbb{R}$. The following are equivalent:

1. K is compact
2. K is closed and bounded
3. Every open cover of K has a finite subcover

3.2.2 Perfect Sets

Definition. A subset $P \subseteq \mathbb{R}$ is perfect if its closed with no isolated points.

Theorem. A non-empty perfect set is uncountable.

3.3 Connected Sets

Definition. Two sets $X, Y \subseteq \mathbb{R}$ are separated if $\bar{X} \cap Y = X \cap \bar{Y} = \emptyset$.
A set $Z \subseteq \mathbb{R}$ is disconnected if $Z = X \cup Y$ where X and Y are separated sets.
A set that is not disconnected is connected.

Examples:

- Disjoint open intervals $(0, 1)$ and $(2, 3)$
- $\mathbb{Q}$ is disconnected

Theorem. A set $Z \subseteq \mathbb{R}$ is connected if, and only if, for all non-empty disjoint sets X and Y satisftying $Z = X \cup Y$, there exists a convergent sequence $x_n \to x$ such that if $(x_n)_{n \in \mathbb{N}} \subseteq X$, then $x \in Y$.

Theorem. A subset $Z \subseteq \mathbb{R}$ is connected if, and only if, it's an interval.

Baire Category Theorem
The intersection of countably many dense open subsets of $\mathbb{R}$ is dense.

Definition. A set is F_σ if it can be written as a countable union of closed sets. A set is G_δ if it can be written as a countable intersection of open sets.

3.4 Cantor Set

Let $C_0 = [0, 1]$ and define

$$C_1 := C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$
$$C_2 := \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$$

and so on, so each C_n is constructed by removing the middle $\frac{1}{3}$ of each component of C_{n-1} for all $n \in \mathbb{N}$. As such, each C_n is equal to 2^n closed intervals of length $\frac{1}{3^n}$.
Define the Cantor set as

$$C = \bigcap_{n \in \mathbb{N}} C_n$$

For all $n \in \mathbb{N}$, $0, 1 \in C_n$, so $0, 1 \in C$.

- If y is an endpoint of one of the intervals of C_n, then y is also an endpoint of an interval of C_{n+1}, so $y \in C_n$ for all n
  - C contains at least the endpoints of all components of all C_n

Additionally, C has 0 length and is uncountable (i.e.: $|C| = |\mathbb{R}|$).

Proposition. C is compact.

Proof. Since $C \subseteq [0, 1]$, C is bounded. Since C_n is a finite intersection of closed sets, each C_n is closed, so since C is the arbitrary intersection of closed sets, then C itself must be closed. By Heine-Borel, C is compact. ■

4 Functional Limits and Continuity

4.1 Functional Limits

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. We say

$$\lim_{x \to c} f(x) = L$$

if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $0 < |x - c| < \delta$, $|f(x) - L| < \varepsilon$.

Proposition. Given $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$, the following are equivalent:

1. $\lim_{x \to c} f(x) = L$
2. For every $(x_n)_{n \in \mathbb{N}}$ with $x_n \neq c$ for all n and $x_n \to c$, $f(x_n) \to L$

Corollary. Let $f, g : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Assume

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = M$$

Then,

1. $\lim_{x \to c}(f+g)(x) = L + M$
2. $\lim_{x \to c}(fg)(x) = LM$
3. If $M \neq 0$, then $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{L}{M}$

4.2 Continuity

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.
A function is continuous on a set $X \subseteq \mathbb{R}$ if the function is continuous at every $a \in X$.

Proposition. For every $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, the following are equivalent:

1. f is continuous at a
2. $\lim_{x \to a} f(x) = f(a)$
3. If $x_n \to a$, then $f(x_n) \to f(a)$

Theorem. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Then

1. $f + g$ is continuous at a
2. fg is continuous at a
3. If $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a

Theorem. Given $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and assuming $f(A) \subseteq B$, if f is continuous at $a \in A$, $g \in f(a) \in B$, then $g \circ f$ is continuous at a.

4.2.1 Continuity on Compact Sets

Theorem. The continuous image of a compact set is compact.

Extreme Value Theorem
If $f : K \to \mathbb{R}$ is continuous on a compact $K \subseteq \mathbb{R}$, then f attains both a maximum and minimum value.

Proof. Since f is continuous, then $f(K)$ is compact, thus closed and bounded by Heine-Borel. By Axiom of Completeness, $\sup(f(K))$ and $\inf(f(K))$ exist, and since $f(K)$ is closed, $\sup(f(K)), \inf(f(K)) \in f(K)$, thus $\max(f(K))$ and $\min(f(K))$ exist, as required. ■

4.2.2 Uniform Continuity

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $X \subseteq \mathbb{R}$ if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $x, y \in X$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

- By order of quantifiers, δ can ONLY depend on ε for uniformly continuous and must be preset to work for *any* $x, y \in \mathbb{R}$

Theorem. A function that is continuous on a compact set K is uniformly continuous on K.

Theorem. A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if there exists $\varepsilon > 0$ and 2 sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in A satisfying

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| \geq \varepsilon$$

4.3 Intermediate Value Theorem

Intermediate Value Theorem
If $f : [a, b] \to \mathbb{R}$ is continuous and L is a number between $f(a)$ and $f(b)$, then there exists some $c \in (a, b)$ such that $L = f(c)$.

Theorem. Let $f : A \to \mathbb{R}$ be continuous. If $E \subseteq A$ is connected, then so is $f(E)$.

5 The Derivative

5.1 Definition of Derivative

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. The derivative of f at c is defined as

$$f'(c) := \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{h\to 0} \frac{f(c + h) - f(c)}{h}$$

If $f'(c)$ exists for all $c \in X$, then f is differentiable on X.

Proposition. Differentiable functions are continuous.

Proof. Let f be differentiable on a set X. Let $c \in X$. Since f is differentiable at c, then

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

By the Algebraic Limit Theorem

$$\lim_{x\to c}(f(x) - f(c)) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}(x - c) = f'(c)(0) = 0$$

since this limit exists. This implies $\lim_{x\to c} f(x) = f(c)$, thus f is continuous at c. ■

Theorem. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions at a point $c \in \mathbb{R}$. Then

1. $(f+g)'(c) = (f'+g')(c)$
2. $(kf)'(c) = kf'(c)$
3. $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$
4. Provided $g(c) \neq 0$, $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$

Theorem (Chain Rule)**.** Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be differentiable functions such that range$(f) \subseteq B$. Then

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

5.1.1 Darboux's Theorem

Theorem (Interior Extreme Value)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on (a, b). If $c \in (a, b)$ is an extremum of f, then $f'(c) = 0$.

Theorem (Darboux)**.** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $[a, b]$ and $z \in \mathbb{R}$ is between $f'(a)$ and $f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = z$.

5.2 Mean Value Theorem

Theorem (Rolle)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Mean Value Theorem
If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define a function $h(x) = f(x)\left(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right)$, so h' exists and h is continuous on $[a, b]$. Since $h(b) = h(a)$, then by Rolle's Theorem, there exists some $c \in (a, b)$ such that $h'(c) = 0$, so

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

as required. ∎

Corollary. If $g : A \to \mathbb{R}$ is differentiable on an interval A and $g'(A) = \{0\}$, then $g(x) = k$ for some $k \in \mathbb{R}$.

Corollary. If f, g are differentiable on an interval A and satisfy $f' = g'$, then $f(x) = g(x) + k$ for some $k \in \mathbb{R}$.

Theorem (Generalized Mean Value)**.** If $f, g : \mathbb{R} \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = f'(c)(g(b) - g(a))$$

6 Sequences and Series of Functions

6.1 Uniform Convergence of a Sequence of Functions

6.1.1 Pointwise Convergence

Definition. For each $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function for some $A \subseteq \mathbb{R}$. The sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise on A to $f : A \to \mathbb{R}$ if for all $x \in A$, $(f_n(x))_{n\in\mathbb{N}}$ converges to x. Equivalently, $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f if

$$\forall x \in A, \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |f_n(x) - f(x)| < \varepsilon \text{ whenever } n \geq N$$

- Notice that with this order of quantifiers, the choice of N can depend on ε and a fixed $x \in A$

6.1.2 Uniform Convergence

Definition. A sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly on a set A to a function f if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$ and $x \in A$. Equivalently, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall x \in A, |f_n(x) - f(x)| < \varepsilon \text{ whenever } n \geq N$$

- Notice that with this order of quantifiers, the choice of N can only depend on ε and must be preset to work for any choice of x, unlike in pointwise convergence where N can be modified to work for any $x \in A$

Cauchy Criterion for Uniform Convergence

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A if, and only if, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $x \in A$, $|f_m(x) - f_n(x)| < \varepsilon$ whenever $m, n \geq N$.

Theorem. The uniform limit of continuous functions is itself continuous. In other words, if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on A to f and all f_n are continuous, then f is itself continuous.

6.2 Uniform Convergence and Differentiation

Theorem. If $f_n \to f$ pointwise on $[a, b]$, each f_n is differentiable and $f_n' \to g$ uniformly on $[a, b]$, then f is differentiable with $f' = g$.

- Basically, if $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $[a, b]$, $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$, then the limit of $(f_n)_{n\in\mathbb{N}}$ is itself differentiable and its derivative is the limit of $(f_n')_{n\in\mathbb{N}}$

Theorem. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions defined on closed interval $[a, b]$ and assume $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$. If there exists some point $x_0 \in [a, b]$ such that $(f_n(x_0))_{n\in\mathbb{N}}$ is convergent, then $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$.

6.3 Series of Functions

Definition. For each $n \in \mathbb{N}$, let f_n and f be functions. The infinite series

$$\sum_{n\in\mathbb{N}} f_n$$

converges pointwise to f if the sequence of partial sums $(s_k)_{k\in\mathbb{N}}$, defined by

$$s_k := \sum_{n\leq k} f_n$$

converges pointwise to f.
If $(s_k)_{k\in\mathbb{N}}$ converges uniformly to f on a set X, then $\sum\limits_{n\in\mathbb{N}} f_n$ converges uniformly to f on X.

Theorem. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions and $\sum\limits_{n\in\mathbb{N}} f_n$ converges uniformly on $X \subseteq \mathbb{R}$, then it is continuous.

Theorem. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions on an interval $[a, b]$, and assume

$$g := \sum_{n\in\mathbb{N}} f_n'$$

converges uniformly. If there exists some $x \in [a, b]$ such that $\sum\limits_{n\in\mathbb{N}} f_n(x)$ converges, then $\sum_{n\in\mathbb{N}} f_n$ converges uniformly to a differentiable function with derivative g on $[a, b]$. Equivalently,

$$\left(\sum_{n\in\mathbb{N}} f_n\right)' = \sum_{n\in\mathbb{N}} f_n'$$

Theorem (Weierstrass M-Test)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions on a set $X \subseteq \mathbb{R}$ and assume there are positive real numbers $(M_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$\sup_{x\in X} |f_n'(x)| < M_n$$

If $\sum_{n\in\mathbb{N}} M_n$ converges, then $\sum_{n\in\mathbb{N}} f_n$ converges uniformly on X.

7 The Riemann Integral

7.1 Definition of the Riemann Integral

7.1.1 Partitions, Upper/Lower Sums

Definition. A partition of $[a, b]$ is a finite, ordered set

$$P = \{a = x_1 < \ldots < x_n = b\}$$

For each subinterval $[x_k, x_{k+1}]$ of P, let

$$m_k := \inf\{f(x) : x \in [x_k, x_{k+1}]\} \qquad M_k := \sup\{f(x) : x \in [x_k, x_{k+1}]\}$$

The lower sum of f with respects to P is

$$L(f, P) := \sum_{k\leq n-1} m_k(x_{k+1} - x_k)$$

The upper sum of f with respects to P is

$$U(f, P) := \sum_{k\leq n-1} M_k(x_{k+1} - x_k)$$

- Clearly, $U(f, P) \geq L(f, P)$

Definition. A partition Q is a refinement of P if $P \subseteq Q$

Lemma. If Q is a refinement of P, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.

Lemma. If P and Q are partitions, then $L(f, P) \leq U(f, Q)$.

7.1.2 Integrability

Definition. Let $P^* = \{\text{partitions of}[a, b]\}$. The upper integral of f is

$$U(f) := \inf\{U(f, P) : P \in P^*\}$$

The lower integral of f is

$$L(f) := \sup\{L(f, P) : P \in P^*\}$$

Lemma. For any bounded f on $[a, b]$, $L(f) \leq U(f)$.

Definition. A bounded function f defined on $[a, b]$ is Riemann integrable if $U(f) = L(f)$, and its integral is defined by

$$\int_a^b f = U(f) = L(f)$$

Theorem (ε-characterization of integrability)**.** A bounded f is integrable over $[a, b]$ if, and only if, for all $\varepsilon > 0$, there exists P_ε of $[a, b]$ suhc that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

Corollary. A continuous function over $[a, b]$ is integrable.

7.2 Properties of the Riemann Integral

Theorem. Let $f : [a, b] \to \mathbb{R}$ be bounded and $c \in (a, b)$. f is integrable if, and only if, f is integrable over $[a, c]$ and $[c, b]$. If so,

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

Theorem. If both f and g are integrable over $[a, b]$, then

1. The function $f + g$ is integrable on $[a, b]$ with

$$\int_a^b f + g dx = \int_a^b f dx + \int_a^b g dx$$

2. For every $c \in \mathbb{R}$, cf is integrable over $[a, b]$ and

$$\int_a^b cf dx = c \int_a^b f dx$$

3. If $f \leq g$, then

$$\int_a^b f dx \leq \int_a^b g dx$$

4. $|f|$ is integrable over $[a, b]$, and

$$\left| \int_a^b f dx \right| \leq \int_a^b |f| dx$$

Theorem. Assume $f_n \to f$ uniformly on $[a, b]$ and each f_n is integrable. Then, f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b f dx$$

7.3 Fundamental Theorem of Calculus

Theorem (FTC 1)**.** If $f : [a, b] \to \mathbb{R}$ is integrable and $F : [a, b] \to \mathbb{R}$ satisfies $F' = f$, then

$$\int_a^b f dx = F(b) - F(a)$$

Theorem (FTC 2)**.** Let $g : [a, b] \to \mathbb{R}$ be integrable and define

$$G(x) = \int_a^x g dy$$

for all $x \in [a, b]$. Then G is continuous on $[a, b]$. If g is continuous at some $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.